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Non-linear convection in a porous medium with inclined temperature gradient and variable gravity effects

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Abstract

The energy method is developed to discuss the non-linear stability of convection in a horizontal porous layer subjected to an inclined temperature gradient and a variable gravity field. Both linear and non-linear stability analyses are carried out for a large number of parameter values. The eigenvalue problems in both cases are solved by the Chebyshev tau-QZ method with optimization routine. It is found that the preferred mode at the onset of convection is a longitudinal mode and that a decrease in gravity variation has a stabilizing effect on the system. Comparisons between the linear and energy stability results show that as the horizontal Rayleigh number increases the difference between the two results increases and thus indicates the possibility of subcritical instability.

1. Introduction

The stability problem of the steady convective flow, which is caused by an inclined temperature gradient in the shallow horizontal layer of the porous medium has been studied by Weber [1], and Nield [2,3]. Weber's analysis was concerned with a small non-dimensional horizontally applied temperature gradient β, and used a perturbation method to solve the eigenvalue problem. Nield's [2] analysis removed the restriction of small β and employed a lower order Galerkin approximation to solve the eigenvalue problem. In a subsequent paper, Nield [3] improved the accuracy of the results of his previous paper by considering a higher order Galerkin approximation and showed the effect of increasing horizontal Rayleigh number on the value of critical vertical Rayleigh number. Kaloni and Qiao [4] presented the non-linear energy stability of this problem and provided more accurate results for both linear and non-linear theories. Further extensions of this problem by considering the effects of adding a net horizontal mass or a vertical through flow have been investigated by Nield [5,6]. The corresponding non-linear results by the energy stability method for these extensions have been reported by Qiao and Kaloni [7,8].

Our purpose here is to generalize the results of [3,4] by considering the variation of the gravity field. We present both the non-linear stability results via the energy method and the linear stability results for the title problem. As is well-known, consideration of variable gravity effect are of likely importance in large-scale convection problems and also in material processing technology. We find that the decreasing gravity parameter has a stabilizing effect on the system. We also note that as the values of longitudinal Rayleigh number increase the gap between the linear and energy stability results widens. We thus have the possibility of a sub-critical instability. We point out that Straughan [9] has studied the convection in a variable gravity field in a viscous fluid layer and Rionero and Straughan [10] in a porous layer with internal heat source subjected to a vertical temperature gradient only. Both these papers employ the energy method, and also provide the linear stability results. In the absence of heat source, the results of [10] agree with our results when we eliminate the consideration of horizontal temperature gradient in our analysis. In the absence of gravity variation, we recover the results of Nield [3] and Kaloni and Qiao [4].

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Nomenclature

g	gravitational acceleration
H	layer height
$\mathbf{i}, \mathbf{j}, \mathbf{k}$	unit vectors in the x-, y- and z-directions, respectively
K	permeability
P, p	dimensionless pressure/perturbed pressure
p_s	dimensionless steady state pressure
R_H	horizontal Rayleigh number
R_V	vertical Rayleigh number
R_L	the critical vertical Rayleigh number of linear theory
R_E	the critical vertical Rayleigh number of energy theory
T	dimensionless temperature
T_s	dimensionless steady state temperature
t	dimensionless time
$\mathbf{u}, \mathbf{v}$	dimensionless perturbed velocity vector/velocity vector
$\mathbf{u}_s$	dimensionless steady state velocity vector
x, y, z	dimensionless Cartesian coordinates

Greek symbols

α	dimensionless overall horizontal wave number
α_x	dimensionless horizontal wave number in x-direction
α_y	dimensionless horizontal wave number in y-direction
α_m	thermal diffusivity
β_T	horizontal temperature gradient
γ_T	coefficient of volume expansion
θ	perturbed dimensionless temperature
κ	thermal diffusivity
μ	dynamic viscosity
ρ_0	density at the reference temperature
ρ_f	the density

We remark that the linear theory gives a sufficient condition for the instability of basic flow whereas the energy theory gives a sufficient condition for the stability of basic flow. In the former case, above the critical Rayleigh number, the system becomes linearly unstable and convective motion begins. It is, however, possible that convection could commence below the critical value of the Rayleigh number and in that case, the energy results become very important. For, they delimit a band of Rayleigh number, where possible sub-critical instabilities might arise. Besides, below the critical value they also guarantee the decay of arbitrary disturbances at all times. We also point out that whenever the energy stability results are close to linear instability results, there is a good possibility of stationary convection. However, wide spectral boundaries between the two methods suggest a possible overstable convection.

2. Governing equations

We investigate the problem of Bénard convection in a fluid-saturated porous medium induced by an inclined temperature gradient with the gravity g depending on the vertical coordinate z^*. The Cartesian axes are chosen with the z^*-axis vertically upwards and the x^*-axis in the direction of applied horizontal temperature gradient β_T. We assume that a porous medium occupies an infinite horizontal layer of height H. The vertical temperature difference across the boundaries is ΔT. It is assumed that the flow in the porous medium is governed by the Darcy's law, which is modified to add the gravitational term to the right-hand side of the Darcy equation. We shall also adopt the Boussinesq approximation which takes the density ρ_f a constant everywhere except in the body force term which can be expressed as

$$\rho_f = \rho_0[1 - \gamma_T(T^* - T_0)], \tag{1}$$

where ρ_0 is a constant, T^* the temperature, T_0 the reference temperature and γ_T is the constant coefficient of volumetric expansion. The variation of gravity g is assumed to be given by

$$g = g(z^*) = g_0 h(z^*/H), \tag{2}$$

where g_0 is the constant gravitational acceleration. For convective motion of an incompressible fluid in a porous medium, the relevant governing equations may be written as [3,4]

$$\nabla^* \cdot \mathbf{v}^* = 0, \tag{3}$$

$$\frac{\mu}{K}\mathbf{v}^* + \nabla^* P^* = \rho_f g \mathbf{k}, \tag{4}$$

$$(\rho c)_m \frac{\partial T^*}{\partial t^*} + (\rho c)_f \mathbf{v}^* \cdot \nabla^* T^* = k_m \nabla^{*^2} T^*, \tag{5}$$

where $\mathbf{v}^* = (u^*, v^*, w^*)$ is the seepage velocity, P^* the pressure, and $\mathbf{k}$ is the unit vector in the z^*-direction. The subscripts m and f refer to the porous medium and the fluid. μ, c, K and k_m denote the viscosity, the specific heat, the permeability of the medium, and the thermal conductivity, respectively. The boundary conditions are

$$w^* = 0, \quad T^* = T_0 - (\pm\Delta T)/2 - \beta_T x^* \quad \text{at } z^* = \pm H/2. \tag{6}$$

We introduce the following non-dimensional quantities

$$\mathbf{x} = \mathbf{x}^*/H, \quad t = \alpha_m t^*/(AH^2), \quad \mathbf{v} = H\mathbf{v}^*/\alpha_m,$$

$$P = K\left(P^* + \rho_0 g_0 \int h(z)\,\mathrm{d}z\right)\Big/(\mu\alpha_{\mathrm{m}}),$$

$$T = R_{\mathrm{V}}(T^* - T_0)/\Delta T,$$

$$\alpha_{\mathrm{m}} = k_{\mathrm{m}}/(\rho_0 c)_{\mathrm{f}}, \quad A = (\rho c)_{\mathrm{m}}/(\rho_0 c)_{\mathrm{f}},$$

$$R_{\mathrm{V}} = \frac{\rho_0 g_0 \gamma_T K H \Delta T}{\mu\alpha_{\mathrm{m}}}, \quad R_{\mathrm{H}} = \frac{\rho_0 g_0 \gamma_T K H^2 \beta_T}{\mu\alpha_{\mathrm{m}}},$$

where R_{V} and R_{H} are referred to as the vertical Rayleigh number and the horizontal Rayleigh number, respectively. Upon substitution of these non-dimensional variables into Eqs. (3)–(5) and with the use of Eqs. (1) and (2), the non-dimensional governing equations then take the form (omitting the star over each variable hereafter for the sake of convenience):

$$\nabla\cdot\mathbf{v} = 0, \tag{7}$$

$$\mathbf{v} + \nabla P = h(z)T\mathbf{k}, \tag{8}$$

$$\frac{\partial T}{\partial t} + (\mathbf{v}\cdot\nabla)T = \nabla^2 T. \tag{9}$$

The corresponding non-dimensional boundary conditions are:

$$w = 0, \quad T = -(\pm R_{\mathrm{V}}/2) - R_{\mathrm{H}}x \quad \text{at } z = \pm 1/2. \tag{10}$$

In the following, we will confine our attention to the case where the gravity field is governed by a linear profile, i.e., $h(z) = 1 - \epsilon(z + (1/2))$ with ϵ being a constant; the analysis, however, is applicable to a wide variety of other fields.

The basic steady state solution $(\mathbf{u}_{\mathrm{s}}, T_{\mathrm{s}}, p_{\mathrm{s}})$ of Eqs. (7)–(9) subjected to boundary conditions (10) is given by

$$u_{\mathrm{s}} = R_{\mathrm{H}}(\beta_0 + \beta_1 z + \beta_2 z^2), \quad v_{\mathrm{s}} = 0, \ \ w_{\mathrm{s}} = 0,$$

$$\nabla T_{\mathrm{s}} = \big(-R_{\mathrm{H}}, 0, -R_{\mathrm{V}} + R_{\mathrm{H}}^2(\gamma_0 + \gamma_1 z + \gamma_2 z^2 + \gamma_3 z^3)\big),$$

$$\nabla p_{\mathrm{s}} = T_{\mathrm{s}} h(z)\mathbf{k} - \mathbf{u}_{\mathrm{s}}, \tag{11}$$

where

$$\delta = 1 - \frac{\epsilon}{2}, \quad \beta_0 = \frac{\epsilon}{24}, \quad \beta_1 = \delta, \quad \beta_2 = -\frac{\epsilon}{2}, \tag{12}$$

$$\gamma_0 = \frac{\delta}{24}, \quad \gamma_1 = -\frac{\epsilon}{24}, \quad \gamma_2 = -\frac{\delta}{2}, \quad \gamma_3 = \frac{\epsilon}{6}, \tag{13}$$

and where we have imposed the requirement that there is no net horizontal mass flux in the x-direction:

$$\int_{-1/2}^{1/2} u_{\mathrm{s}}\,\mathrm{d}z = 0. \tag{14}$$

We now perturb the steady state solution as follows:

$$\mathbf{v} = \mathbf{u}_{\mathrm{s}} + \mathbf{u}, \quad T = T_{\mathrm{s}} + \theta, \quad P = p_{\mathrm{s}} + p. \tag{15}$$

The perturbation equations then take the form

$$\nabla\cdot\mathbf{u} = 0, \tag{16}$$

$$\mathbf{u} + \nabla p = \theta h(z)\mathbf{k}, \tag{17}$$

$$\frac{\partial\theta}{\partial t} + (\mathbf{u}\cdot\nabla)\theta = \nabla^2\theta - \mathbf{u}_{\mathrm{s}}\cdot\nabla\theta - \mathbf{u}\cdot\nabla T_{\mathrm{s}}, \tag{18}$$

where $\mathbf{u}_{\mathrm{s}}$ and ∇T_{s} are given by (11). The corresponding boundary conditions become

$$w = \theta = 0 \quad \text{at } z = \pm 1/2. \tag{19}$$

3. The eigenvalue problems of non-linear and linear stability analysis

As the linearized system of Eqs. (16)–(18) is not symmetric (self-adjoint), the non-linear energy method will thus give different results from the linear stability method. To perform the non-linear energy stability analysis, we define an energy functional as

$$E(t) = \frac{\xi}{2}\|\theta\|^2, \tag{20}$$

where ξ is the positive coupling parameter. On multiplying Eq. (17) by **u**, Eq. (18) by θ and the integrating over V, we find (after using the boundary conditions and divergence theorem)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|^2 = -\|\nabla\theta\|^2 - \langle(\mathbf{u}\cdot\nabla T_s)\theta\rangle, \tag{21}$$

$$\|\mathbf{u}\|^2 = \langle h(z)\theta w\rangle. \tag{22}$$

Here, V denotes a typical periodicity cell, $\langle\cdot\rangle$ denotes the integration over V, and $\|\cdot\|$ denotes the L^2(V) norm. The system of Eqs. (21) and (22), along with (20), can be put in the form

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \mathscr{I} - \mathscr{D}, \tag{23}$$

where

$$\mathscr{I} = -\xi\langle(\mathbf{u}\cdot\nabla T_{\mathrm{s}})\theta\rangle + \langle h(z)\theta w\rangle, \tag{24}$$

$$\mathscr{D} = \xi\|\nabla\theta\|^2 + \|\mathbf{u}\|^2. \tag{25}$$

We now define

$$m = \max_{\mathscr{H}} \frac{\mathscr{I}}{\mathscr{D}}, \tag{26}$$

where $\mathscr{H}$ is the space of admissible solutions. On combining (23) with (25) and (26), and by using Poincaré inequality, we can infer, for $0 < m < 1$, that

$$\frac{\mathrm{d}E}{\mathrm{d}t} \leqslant -2\pi^2(1-m)E. \tag{27}$$

Inequality (27) clearly indicates that for $0 < m < 1$, $E(t) \to 0$ at least exponentially as $t \to \infty$.

We now return to (26) and consider the maximum problem at the critical argument $m = 1$. The associated Euler–Lagrange equations become

$$-\xi \nabla T_s \cdot \mathbf{u} + hw + 2\xi \nabla^2 \theta = 0, \tag{28}$$

$$\xi \nabla T_s \theta - h\theta \mathbf{k} + 2\mathbf{u} = \nabla \omega, \tag{29}$$

where ω is a Lagrange multiplier introduced since $\mathbf{u}$ is solenoidal. We perform the following standard normal mode analysis

$$[u, v, w, \theta, \omega] = [u(z), v(z), w(z), \theta(z), \omega(z)] \exp[\mathrm{i}(\alpha_x x + \alpha_y y)]. \tag{30}$$

On eliminating the variables u, v, and ω, we derive the corresponding eigenvalue equations, which can, after some rearrangement of terms, be written as

$$\frac{2\lambda}{\alpha^2}(D^2 - \alpha^2)w + (\lambda h - f)\theta + \mathrm{i}\frac{\alpha_x R_\mathrm{H}}{\alpha^2} D\theta = -R_\mathrm{V}\theta, \tag{31}$$

$$2\left(D^2 - \alpha^2 + \frac{\alpha_y^2 R_\mathrm{H}^2}{4\lambda\alpha^2}\right)\theta + (\lambda h - f)w + \mathrm{i}\frac{\alpha_x R_\mathrm{H}}{\alpha^2} Dw = -R_\mathrm{V} w, \tag{32}$$

where $\lambda = \xi^{-1}$, $\alpha^2 = \alpha_x^2 + \alpha_y^2$, $D^j = \mathrm{d}^j/\mathrm{d}z^j$ $(j = 1, 2)$, and

$$f = f(z) = R_\mathrm{H}^2(\gamma_0 + \gamma_1 z + \gamma_2 z^2 + \gamma_3 z^3). \tag{33}$$

For the purpose of comparison, we will also carry out the numerical calculation based on the linear stability theory in the next section. To this end, the corresponding eigenvalue equations of linear stability analysis are readily derived from Eqs. (16)–(18) by performing the mode analysis

$$[u, v, w, \theta, \omega] = [u(z), v(z), w(z), \theta(z), \omega(z)] \times \exp\left[\mathrm{i}(\alpha_x x + \alpha_y y) + \sigma t\right]. \tag{34}$$

We, thus obtain

$$\mathscr{L}_1(w, \theta) \equiv (D^2 - \alpha^2)w + \alpha^2 h\theta = 0, \tag{35}$$

$$\mathscr{L}_2(w, \theta) \equiv (D^2 - \alpha^2 - \mathrm{i}\alpha_x u_s)\theta + \mathrm{i}\frac{\alpha_x}{\alpha^2} R_\mathrm{H} Dw - DT_s w - \sigma\theta = 0, \tag{36}$$

where σ is the time growth rate. Both eigenvalue equations, Eqs. (35) and (36) of linear stability analysis, and Eqs. (31) and (32) of non-linear energy stability analysis, are subjected to the boundary conditions

$$w = \theta = 0 \quad \text{at } z = \pm 1/2. \tag{37}$$

4. Numerical solutions via Chebyshev tau-QZ method

To solve the eigenvalue problems (35)–(37), and (31), (32) and (37), we employ the Chebyshev tau-QZ method. We first consider Eqs. (35)–(37). To this end, we turn to solve the following equations

$$\mathscr{L}_j(w, \theta) = \tau_{2j-1} T_{N+1}(z) + \tau_{2j} T_{N+2}(z), \quad j = 1, 2. \tag{38}$$

We expand the solution of (38) in a finite series of Chebyshev polynomials as

$$(w, \theta) = \sum_{k=0}^{N+2} (w_k, \theta_k) T_k(z), \tag{39}$$

where $T_k(z)$ is the Chebyshev polynomial of degree k defined by

$$T_k(z) = \cos(k \arccos z), \quad -1 \leqslant z \leqslant 1, \ k = 0, 1, \ldots \tag{40}$$

We reset the domain from $[-1/2, 1/2]$ to $[-1, 1]$ with the coordinate transformation of z to $2z$, and introduce the inner product defined in the weighted space $L^2(-1, 1)$ as

$$\langle r, s \rangle = \int_{-1}^{1} \frac{rs}{\sqrt{(1 - z^2)}} \, \mathrm{d}z. \tag{41}$$

Since the Chebyshev polymials are orthogonal in the space defined above, we obtain $2(N + 1)$ equations

$$\langle \mathscr{L}_j(w, \theta), T_k \rangle = 0, \quad k = 0, 1, \ldots, N, \ j = 1, 2. \tag{42}$$

There are four further conditions arising from (42) and these are:

$$\langle \mathscr{L}_j(w, \theta), T_{N+k} \rangle = \tau_{2(j-1)+k} \|T_{N+k}\|^2, \quad k = 1, 2, \ j = 1, 2 \tag{43}$$

and these may be effectively used to calculate the $\tau's$, which in turn may be used to measure the error associated with the truncation of an infinite series. Here, $\|\cdot\|$ is the norm associated with the inner product (41). The boundary conditions (37) provide us the other four equations, i.e.,

$$\sum_{j=0}^{N+2} (-1)^j w_j = 0, \quad \sum_{j=0}^{N+2} w_j = 0, \tag{44}$$

$$\sum_{j=0}^{N+2} (-1)^j \theta_j = 0, \quad \sum_{j=0}^{N+2} \theta_j = 0. \tag{45}$$

The eigenvalue equations (42) subjected to (44) and (45) could be written in the matrix form as

$$\mathbf{A}_1 \mathbf{x} = \sigma \mathbf{B}_1 \mathbf{x}, \tag{46}$$

where $\mathbf{x} = (w_0, w_1, \ldots, w_{N+2}, \theta_0, \theta_1, \ldots, \theta_{N+2})$, and

$$\mathbf{A}_1 = \begin{bmatrix} 4\mathbf{D}^2 - \boldsymbol{\alpha}^2\mathbf{I} & \mathbf{H}_1 \\ \mathbf{H}_2 & 4\mathbf{D}^2 - \boldsymbol{\alpha}^2\mathbf{I} + \mathbf{H}_3 \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix},$$

and where the matrices $\mathbf{0}, \mathbf{I}$ are zero and identity matrices of $N+3$ by $N+3$. Here,

$$\begin{aligned}
&\mathbf{D} = [D_{i,j}] = \frac{2}{\pi c_i}[\langle T_j', T_i\rangle] \quad i,j = 0,1,\ldots,N+2,\\
&\mathbf{D}^2 = [D_{i,j}^2] = \frac{2}{\pi c_i}[\langle T_j'', T_i\rangle] \quad i,j = 0,1,\ldots,N+2,\\
&\mathbf{H}_1 = \alpha^2(\delta\mathbf{I} + \beta_2\mathbf{Z}), \qquad (47)\\
&\mathbf{H}_2 = (R_\mathrm{V} - R_\mathrm{H}^2\gamma_0)\mathbf{I} - R_\mathrm{H}^2\left(\frac{\gamma_1}{2}\mathbf{Z} + \frac{\gamma_2}{4}\mathbf{Z}^2 + \frac{\gamma_3}{8}\mathbf{Z}^3\right) + 2\mathrm{i}R_\mathrm{H}\frac{\alpha_x}{\alpha^2}\mathbf{D},\\
&\mathbf{H}_3 = -\mathrm{i}\alpha_x R_\mathrm{H}\left(\beta_0\mathbf{I} + \frac{\beta_1}{2}\mathbf{Z} + \frac{\beta_2}{4}\mathbf{Z}^2\right),
\end{aligned}$$

and in which

$$\begin{aligned}
&\mathbf{Z}^\mathbf{m} = \left[z_{i,j}^{(m)}\right] = \frac{2}{\pi c_i}\left[\langle z^m T_j, T_i\rangle\right]\\
&i,j = 0,1,\ldots,N+2, \quad m = 1,2,3.
\end{aligned} \qquad (48)$$

with the constants c_n given by

$$\begin{cases} c_0 = 2, \\ c_n = 1, & \text{if} \quad n > 0, \\ c_n = 0, & \text{if} \quad n < 0 \end{cases}$$

and the $N+2,\ N+3,\ 2N+5,\ 2N+6$'s row of Eq. (46) are overwritten by the boundary conditions (44) and (45). The critical vertical Rayleigh number R_L is defined by

$$R_\mathrm{L} = \min_{\alpha_x} \min_{\alpha_y} R_\mathrm{V}(\alpha_x, \alpha_y, \epsilon, R_\mathrm{H}) \qquad (49)$$

such that the choice of R_V makes the real part of the leading eigenvalue, σ, of (46), defined as having the largest real part, approaching zero.

Similarly, the generalized eigenvalue equations corresponding to Eqs. (31) and (32) subjected to (37) for the non-linear energy stability analysis could be written as

$$\mathbf{A}_2\mathbf{x} = R_\mathrm{V}\mathbf{B}_2\mathbf{x}, \qquad (50)$$

where

$$\mathbf{A}_2 = \begin{bmatrix} \frac{2\lambda}{\alpha^2}(4\mathbf{D}^2 - \boldsymbol{\alpha}^2\mathbf{I}) & \mathbf{H}_4 \\ \mathbf{H}_4 & 2(4\mathbf{D}^2 - \boldsymbol{\alpha}^2\mathbf{I}) + \mathbf{H}_5 \end{bmatrix},$$

$$\mathbf{B}_2 = \begin{bmatrix} \mathbf{0} & -\mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix},$$

and where

$$\begin{aligned}
\mathbf{H}_4 &= (\lambda\delta - R_\mathrm{H}^2\gamma_0)\mathbf{I} - \frac{\lambda\epsilon + R_\mathrm{H}^2\gamma_1}{2}\mathbf{Z} - \frac{R_\mathrm{H}^2\gamma_2}{4}\mathbf{Z}^2\\
&\quad - \frac{R_\mathrm{H}^2\gamma_3}{8}\mathbf{Z}^3 + i\frac{2\alpha_x R_\mathrm{H}}{\alpha^2}\mathbf{D},\\
\mathbf{H}_5 &= \frac{\alpha_y^2 R_\mathrm{H}^2}{2\lambda\alpha^2}\mathbf{I},
\end{aligned} \qquad (51)$$

The critical vertical Rayleigh number R_E is defined as

$$R_\mathrm{E} = \max_{\lambda} \min_{\alpha_x} \min_{\alpha_y} R_\mathrm{V}(\lambda, \alpha_x, \alpha_y, \epsilon, R_\mathrm{H}), \qquad (52)$$

where R_V is the leading eigenvalue of (50), defined as having the smallest positive eigenvalue. We remark that all the eigenvalues of (50) are real which arise from (31) and (32). This can be shown by multiplying (31) by w^* (complex conjugate), (32) by θ^* and integrating over $(-\frac{1}{2}, \frac{1}{2})$, and then adding the results. Finally taking the imaginary part, we obtain

$$[\langle\theta w^*\rangle + \langle w\theta^*\rangle]R_\mathrm{V}^i = 0, \qquad (53)$$

where $\langle\cdot\rangle$ denotes the integration over $(-\frac{1}{2}, \frac{1}{2})$, and $R_\mathrm{V} = R_\mathrm{V}^r + \mathrm{i}R_\mathrm{V}^i$. The coefficient of R_V^i in (52) is non-zero real unless $w_\mathrm{r}, \theta_\mathrm{r}$ and $w_\mathrm{i}, \theta_\mathrm{i}$ (the real and imaginary parts of w and θ) are orthogonal. Thus, we can conclude that R_V must be real.

To determine the linear stability threshold (49), we first employ the ZGEGV routine of LAPACK library [11], implementing the QZ algorithm of Moler and Stewart [12] for solving the generalized eigenvalue equation (46), for the fixed values of all parameters. Then, the secant method is applied to search for R_V such that the real part of the leading eigenvalue σ approaches zero. Finally, the critical vertical Rayleigh number R_L of (49) is optimized, by using the quasi-Newton algorithm implemented in the routine E04JYF of NAG library, to find the minimum R_L over the horizontal wave numbers α_x and α_y.

In order to find the energy stability threshold (52), for fixed λ, α_x and α_y, ϵ and R_H, we first find the leading eigenvalue of (50) using the QZ algorithm. The critical vertical Rayleigh number R_E of (52) is then solved using the golden section search routine to find the minimun over α_x and α_y, and then the maximun over λ.

5. Results and discussion

We begin our remarks with the important observation that in both the schemes for calculating the energy and linear stability thresholds, we find the onset of instability is with $\alpha_x = 0$, and $\sigma_i = 0$ in the linear case. That is, the preferred mode of disturbance is the longitudinal mode, and the longitudinal stationary mode in the linear case.

Figs. 1 and 2 show the variation of R_L (the critical Rayleigh number in the linear theory) and R_E (the critical Rayleigh number in the energy method) with the horizontal Rayleigh number R_H for two values of the gravity parameter ϵ. From the graphs, we observe that as R_H increases the values of both R_L and R_E increase and thus indicate the stabilizing effect. We note that the values of R_L are always higher than those of R_E. This is quite understandable because the linear stability theory

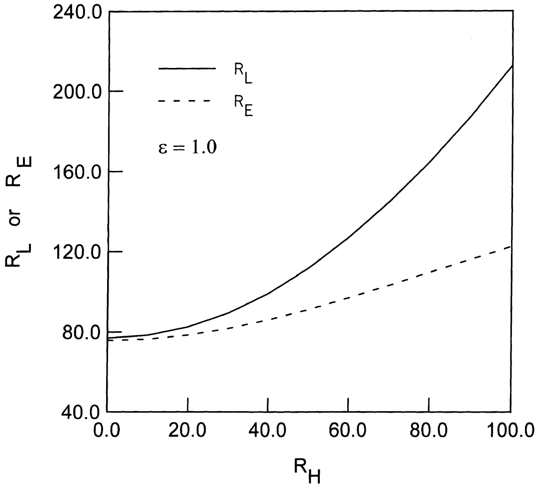

Fig. 1. Linear and energy stability results for $\epsilon = 1.00$.

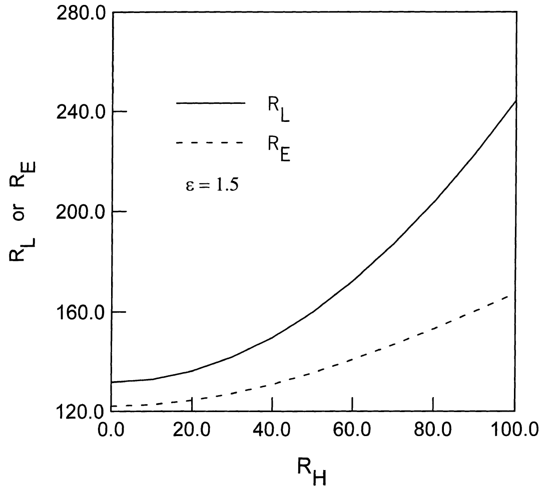

Fig. 2. Linear and energy stability results for $\epsilon = 1.50$.

gives sufficient conditions for instability while the energy stability theory gives the sufficient conditions for stability. Thus, the difference between the values of R_{L} and R_{E} reveals that there is a band of Rayleigh numbers where subcritical instabilities may arise. We note that this band widens as R_{H} increases. In comparison to the results of [3] and [4], we also conclude that a decrease in gravity has a stabilizing effect. This conclusion is further strengthened in Fig. 3 which shows the variation of R_{E} and R_{L} with the gravity parameter ϵ at $R_{\mathrm{H}} = 20$. Here, clearly an increase in ϵ (decrease in gravity) indicates an increase in the R_{L} and R_{E} values.

In order to check whether an increase in the horizontal Rayleigh number values always has a stabilizing effect, we have calculated the R_{L} and R_{E} values for a broad range of values of R_{H} at $\epsilon = 1$. These calculations are reported in Table 1. It is clear from the table that,

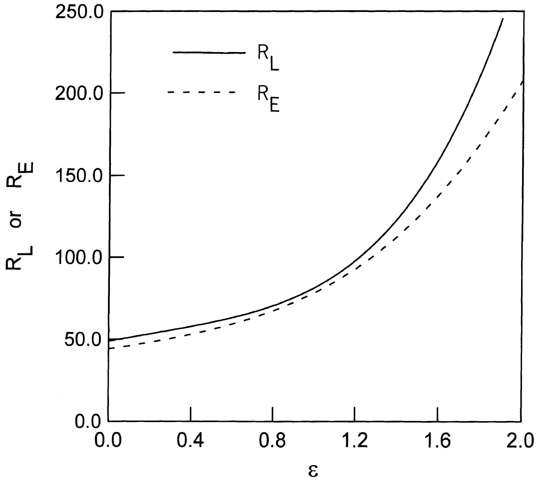

Fig. 3. Variation of R_{L} and R_{E} with gravity parameter ϵ at $R_{\mathrm{H}} = 20$.

initially the values of both R_{L} and R_{E} increase as R_{H} increases. However, as R_{H} continues to increase both R_{L} and R_{E} reach a maximun, and decrease with further increase of R_{H}, and eventually both pass through zero. This means that the horizontal flow becomes unstable even in the absence of an applied variation gradient.

Table 1
R_{L} and R_{E} variations with R_{H} for $\epsilon = 1.0$

R_{H}	R_{L}	R_{H}	R_{E}
0.0000	77.0797	0.0000	75.725
10.000	78.4810	10.000	76.427
20.000	82.6794	20.000	78.491
30.000	89.6575	30.000	81.795
40.000	99.3865	40.000	86.156
50.000	111.826	50.000	91.350
60.000	126.923	60.000	97.142
70.000	144.612	70.000	103.312
80.000	164.811	80.000	109.666
90.000	187.422	90.000	116.038
100.00	212.318	100.00	122.286
110.00	239.339	110.00	128.291
120.00	268.259	120.00	133.943
130.00	298.748	130.00	139.139
140.00	330.221	140.00	143.780
150.00	328.999	150.00	147.761
160.00	327.917	160.00	150.969
170.00	320.415	170.00	153.277
180.00	306.339	180.00	179.726
190.00	285.489	185.00	174.431
200.00	257.745	190.00	163.949
210.00	223.096	195.00	150.232
220.00	181.647	200.00	132.775
230.00	133.578	208.00	102.352
240.00	79.1153	216.00	65.7016
250.00	18.5008	224.00	25.2835
252.87	0.00000	228.20	1.9174

Thus, we conclude that the effect of reducing gravity is to delay the instability of Hadley flow but the reduction in gravity does not eliminate the horizontal flow instability.

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